Maxwell equations and electromagnetic potentials

$$\nabla \times \vec{B} - \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E} = \mu_0 \vec{j} \qquad (1)$$

$$\nabla \times \vec{E} + \frac{\partial}{\partial t}\vec{B} = 0 \qquad (2)$$

$$\nabla \cdot \vec{E} = \frac{\varrho}{\varepsilon_0} \qquad (3)$$

$$\nabla \cdot \vec{B} = 0 \qquad (4)$$

Potentials

$$\boxed{\vec{B} = \nabla \times \vec{A}} \quad \rightarrow \text{ statisfies Eq. (4)}$$

Eq. (2)

$$\nabla \times \left(\vec{E} + \frac{\partial}{\partial t}\vec{A}\right) = 0$$

$$\nabla \times \nabla \phi = 0 \quad \Rightarrow \quad \vec{E} + \frac{\partial}{\partial t}\vec{A} = -\nabla\phi$$

$$\boxed{\vec{E} = -\left(\frac{\partial}{\partial t}\vec{A} + \nabla\phi\right)}$$

Eq. (3)

$$-\frac{\partial}{\partial t}\nabla \cdot \vec{A} - \nabla \cdot \nabla\phi = \frac{\varrho}{\varepsilon_0}$$

$$\Delta\phi = \nabla^2\phi$$

Eq. (1)

$$\nabla \times \nabla \times \vec{A} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} + \frac{1}{c^2}\frac{\partial}{\partial t}\nabla\phi = -\frac{1}{\varepsilon_0 c^2}\vec{j}$$

$$\nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2\vec{A}$$

$$\boxed{\varepsilon_0 \mu_0 = \frac{1}{c^2}}$$

Transform

$$\vec{A} \rightarrow \vec{A} + \nabla \chi$$

arbitrary scalar field

$$\phi \rightarrow \phi - \frac{\partial}{\partial t}\chi$$

The same fields:

$$\vec{E} = -\frac{\partial}{\partial t}\vec{A} \cancel{- \frac{\partial}{\partial t}\nabla\chi} - \nabla\phi \cancel{+ \nabla\left(\frac{\partial}{\partial t}\chi\right)}$$

$$\vec{B} = \nabla\times\vec{A} + \nabla\times\nabla\chi = \nabla\times\vec{A}$$

Coulomb gauge !!!

$$\boxed{\nabla\cdot\vec{A} = 0}$$

Maxwell equations (3) and (1)

(3) $$\Delta\phi = \boxed{\nabla^2\phi = -\frac{\rho}{\varepsilon_0}}$$ Poisson equation

(1) $$\boxed{\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} = -\frac{1}{\varepsilon_0 c^2}\left(\vec{j} - \varepsilon_0\nabla\left(\frac{\partial\phi}{\partial t}\right)\right)}$$

Transverse and longitudino fields

$\vec{a}(\vec{r})$ ... Helmholtz theorem

$$\vec{a}(\vec{r}) = \vec{a}^{\parallel} + \vec{a}^{\perp}$$

$\vec{a}^\perp$ has zero divergence

$$\nabla\cdot\vec{a}^\perp = 0$$

$\vec{a}^\parallel$ has zero rotation

$$\nabla\times\vec{a}^\parallel = 0$$

Magnetic field is purely transverse

$$\vec{B} = \vec{B}^\perp$$

In Coulomb gauge

$$\vec{A} = \vec{A}^\perp$$

Electric field?

Eq. (2) and (3)

$$\nabla\times\vec{E}^\perp = -\frac{\partial}{\partial t}\vec{B}$$

$$\nabla\cdot\vec{E}^\parallel = -\frac{\rho}{\varepsilon_0}$$

Eq. (1)

$$\nabla\times\vec{B} = \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E}^\perp + \mu_0\vec{j}^\perp \qquad (*)$$

$$0 = \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E}^\parallel + \frac{1}{c^2\varepsilon_0}\vec{j}^\parallel \qquad (**) \quad \nabla\cdot$$

$$\frac{1}{\varepsilon_0}\frac{\partial}{\partial t}\rho + \frac{1}{\varepsilon_0}\nabla\cdot\vec{j}^\parallel = 0 \Rightarrow \boxed{\nabla\cdot\vec{j}^\parallel + \frac{\partial}{\partial t}\rho = 0}$$

Continuity equation

= conservation of charge

Time derivative of (*)

using $-\frac{\partial}{\partial t}\vec{B} = \nabla\times\vec{E}^{\perp}$ Eq(2)

$$-\nabla\times\nabla\times\vec{E}^{\perp} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}^{\perp} + \mu_0\frac{\partial}{\partial t}\vec{j}^{\perp}$$

$$\nabla(\underbrace{\nabla\cdot\vec{E}^{\perp}}_{=0}) - \nabla^2\vec{E}^{\perp}$$

$$\Rightarrow \quad \boxed{\nabla^2\vec{E}^{\perp} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}^{\perp} = \frac{1}{\varepsilon_0 c^2}\frac{\partial}{\partial t}\vec{j}^{\perp}}$$

What is $\vec{j}^{\perp}$?

$$\underbrace{\vec{E}^{\perp} = -\frac{\partial}{\partial t}\vec{A}}_{\text{in any gauge}} \quad \text{because} \quad \underbrace{\nabla\cdot\nabla\phi = 0}_{\text{always}}$$

$$-\nabla^2\frac{\partial}{\partial t}\vec{A} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial t}\vec{A} = -\frac{1}{\varepsilon_0 c^2}\frac{\partial}{\partial t}\vec{j}^{\perp}$$

$$\Rightarrow \quad -\nabla^2\vec{A} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} = -\frac{1}{\varepsilon_0 c^2}\vec{j}^{\perp}$$

Eq. (1)

$$\boxed{\vec{j}^{\perp} = \vec{j} - \varepsilon_0\nabla\left(\frac{\partial\phi}{\partial t}\right)}$$

$$\boxed{\vec{j}^{\parallel} = \varepsilon_0\nabla\left(\frac{\partial\phi}{\partial t}\right)}$$

$$\boxed{\vec{E}^{\parallel} = -\nabla\phi}$$

Linear polarization and absorption

$$\nabla^2\vec{E}^{\perp} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}^{\perp} = \frac{1}{\varepsilon_0 c^2}\frac{\partial}{\partial t}\vec{j}^{\perp}$$

$$\vec{j}^{\perp} = \langle q\dot{\vec{r}}\rangle_{space} = \frac{d}{dt}\langle q\vec{r}\rangle = \frac{\partial}{\partial t}\vec{P}(\vec{r})$$

polarization

$$\nabla^2\vec{E}^{\perp}(\vec{r}) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E}^{\perp}(\vec{r}) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\frac{1}{\varepsilon_0}\vec{P}(\vec{r},t)$$

Fourier transform $t \to \omega$, $\vec{r} \to \vec{k}$

$$\boxed{\vec{P}(\omega) = \varepsilon_0\chi(\omega)\vec{E}(\omega)}$$

susceptibility linear (first order)

$$-k^2\vec{E}^{\perp}(\omega) + \frac{\omega^2}{c^2}\vec{E}^{\perp}(\omega) = -\frac{\omega^2}{c^2}\chi(\omega)\vec{E}^{\perp}(\omega)$$

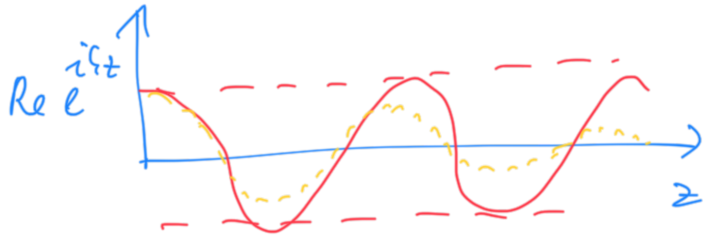
$$\underbrace{\left[-k^2 + \frac{\omega^2}{c^2}\left(1+\chi(\omega)\right)\right]}_{=0}\vec{E}^{\perp}(\omega) = 0$$

$$k^2 = \frac{\omega^2}{c^2}\underbrace{\left(1+\chi(\omega)\right)}_{\varepsilon_r} \quad \Rightarrow \quad k = \frac{\omega}{c}\sqrt{\varepsilon_r} = \frac{\omega}{c}(\underset{\uparrow}{n}+i\underset{\uparrow}{\kappa})$$

real, im of $\sqrt{\varepsilon_r}$

$$\vec{k} = (0,0,k)$$

$$e^{ikz} \xrightarrow{\hspace{3cm}} z$$



$$E \sim e^{ikz} = e^{i\frac{\omega}{c}nz}\, e^{-\frac{\omega}{c}\kappa z}$$

absorption coefficient:

$$I = I_0 e^{-\alpha z} = I_0 e^{-\frac{2\omega\kappa}{c}z}$$

$$\boxed{\alpha = \frac{2\omega}{c}\kappa}$$

$$\varepsilon_r = 1+\chi(\omega) = 1+\underset{\uparrow\,\text{real}}{\chi'(\omega)} + i\underset{\uparrow\,\text{imaginary}}{\chi''(\omega)}$$

$$\sqrt{\varepsilon_r} = \sqrt{1+\chi'+i\chi''} = n+i\kappa \quad /^2$$

$$1+\chi'+i\chi'' = (n+i\kappa)(n+i\kappa) = n^2-\kappa^2+2in\kappa$$

Real $\quad 1+\chi' = n^2 - \kappa^2$

Im $\quad i\chi'' = 2in\kappa \implies$

$$\boxed{\kappa = \frac{\chi''(\omega)}{2n}}$$

$$1+\chi'(\omega) = n^2 + \underbrace{\left(\frac{\chi''(\omega)}{2n}\right)^2}_{\approx 0} \implies \boxed{n = \sqrt{1+\chi'(\omega)}}$$

$$\boxed{\alpha(\omega) = \frac{\omega}{nc}\chi''(\omega)}$$