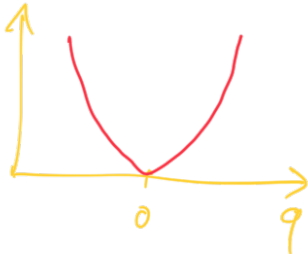
Harmonický oscilátor

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2$$



Hamiltonovy rovnice

$$\left.\begin{aligned} \dot{p} &= -\frac{\partial H}{\partial q} = m\omega^2 q \\ \dot{q} &= \frac{\partial H}{\partial p} = \frac{1}{m} p \end{aligned}\right\} \quad \begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} 0 & -m\omega^2 \\ \frac{1}{m} & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$$

$$M = \begin{pmatrix} 0 & -a \\ b & 0 \end{pmatrix}$$

$$\frac{\omega}{2}\tilde{q}^2 = \frac{m\omega^2}{2} q^2 \quad \Rightarrow \quad q^2 = \frac{\omega}{m\omega^2}\tilde{q}^2$$

$$\Rightarrow q = \frac{1}{\sqrt{m\omega}}\tilde{q}$$

$$\frac{\omega}{2}\tilde{p}^2 = \frac{p^2}{2m}$$

$$p^2 = m\omega\tilde{p}^2$$

$$\Rightarrow p = \sqrt{m\omega}\,\tilde{p}$$

$$\boxed{H = \frac{\omega}{2}\left(\tilde{p}^2 + \tilde{q}^2\right)}$$

$$\left.\begin{aligned} \dot{\tilde{p}} &= -\frac{\partial H}{\partial \tilde{p}} = -\omega\tilde{q} \\ \dot{\tilde{q}} &= \frac{\partial H}{\partial \tilde{p}} = \omega\tilde{p} \end{aligned}\right\} \quad \begin{pmatrix} \dot{\tilde{p}} \\ \dot{\tilde{q}} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} \tilde{p} \\ \tilde{q} \end{pmatrix}}_{\vec{v}}$$

$$S^{-1}\dot{\vec{v}} = S^{-1} A S \; S^{-1} v$$

$$\begin{pmatrix} \dot\alpha \\ \dot\alpha' \end{pmatrix} = \begin{pmatrix} \bullet & 0 \\ 0 & \bullet \end{pmatrix} \begin{pmatrix} \alpha \\ \alpha' \end{pmatrix} \Rightarrow \begin{matrix} \dot\alpha = \bullet\,\alpha \\ \dot\alpha' = \bullet\,\alpha' \end{matrix}$$

$$s_1 = \frac{1}{\sqrt2}\begin{pmatrix} i \\ 1 \end{pmatrix} \; ; \; s_2 = \frac{1}{\sqrt2}\begin{pmatrix} -i \\ 1 \end{pmatrix}$$

$$S = \left(s_1 \,\vdots\, s_2 \right) = \frac{1}{\sqrt2}\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \qquad S^{-1} = S^{\dagger} = \frac{1}{\sqrt2}\begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix}$$

$$A' = \frac12 \begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix} \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} =$$

$$= \frac12 \begin{pmatrix} \omega & +i\omega \\ \omega & -i\omega \end{pmatrix} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} = \frac12 \begin{pmatrix} 2i\omega & 0 \\ 0 & -2i\omega \end{pmatrix}$$

$$= \begin{pmatrix} i\omega & 0 \\ 0 & -i\omega \end{pmatrix}$$

$$\Rightarrow \frac12 \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix} = \frac12 \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\vec{w} = S^{-1} \vec{v}$$

$$\begin{pmatrix} \alpha \\ \alpha' \end{pmatrix} = \frac{1}{\sqrt2} \begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix} \begin{pmatrix} \tilde p \\ \tilde q \end{pmatrix} \Rightarrow \begin{matrix} \alpha = \frac{1}{\sqrt2}(\tilde q - i \tilde p) \\ \alpha' = \frac{1}{\sqrt2}(\tilde q + i \tilde p) \end{matrix}$$

Rešeni !

$$\begin{pmatrix} \dot\alpha \\ \dot\alpha' \end{pmatrix} = \begin{pmatrix} i\omega & 0 \\ 0 & -i\omega \end{pmatrix} \begin{pmatrix} \alpha \\ \alpha' \end{pmatrix} \Rightarrow \begin{matrix} \dot\alpha = i\omega\alpha & \Rightarrow \alpha(t) = e^{i\omega t}\alpha(0) \\ \dot\alpha' = -i\omega\alpha' & \alpha'(t) = e^{-i\omega t}\alpha'(0) \end{matrix}$$

$$a(t) = \frac{1}{\sqrt{2}}(\tilde{q} + i\tilde{p}) \quad ; \quad a(t) = e^{-i\omega t} a(0)$$

$$a^*(t) = \frac{1}{\sqrt{2}}(\tilde{q} - i\tilde{p}) \quad ; \quad a^*(t) = e^{i\omega t} a^*(0)$$

$$\boxed{\tilde{q} = \frac{1}{\sqrt{2}}\left(a(t) + a^*(t)\right)} \quad ; \quad \boxed{\tilde{p} = -\frac{i}{\sqrt{2}}(a - a^*)}$$