Kvantový harmonický oscilátor

$$\hat{H} = \frac{\omega}{2}(\hat{p}^2 + \hat{q}^2) \qquad [\hat{p}, \hat{q}] = -i\hbar$$

$$\hat{a} = \frac{1}{\sqrt{2}}(\hat{q} + i\hat{p})$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}(\hat{q} - i\hat{p})$$

$$[\hat{a}^\dagger, \hat{a}] = \hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger = \frac{1}{2}\left[(q - ip)(q + ip) - (q + ip)(q - ip)\right]$$

$$= \frac{1}{2}\left(\cancel{qq} - ipq + iqp + \cancel{pp} - (\cancel{qq} + ipq - iqp + \cancel{pp})\right)$$

$$= -i(pq - qp) = -i(-i)\hbar = -\hbar$$

$$\boxed{\begin{aligned}\hat{a} &= \frac{1}{\sqrt{2\hbar}}(\hat{q} + i\hat{p}) \\ \hat{a}^\dagger &= \frac{1}{\sqrt{2\hbar}}(\hat{q} - i\hat{p})\end{aligned}} \Rightarrow \boxed{[\hat{a}^\dagger, \hat{a}] = -1}$$

Inverzní vztah

$$\boxed{\begin{aligned}\hat{q} &= \sqrt{\frac{\hbar}{2}}(\hat{a} + \hat{a}^\dagger) \\ \hat{p} &= i\sqrt{\frac{\hbar}{2}}(\hat{a}^\dagger - \hat{a})\end{aligned}}$$

Hamiltonián v $\hat{a}, \hat{a}^\dagger$ $\qquad a^\dagger \equiv a^\dagger$

$$\hat{H} = \frac{\omega}{2}(\hat{p}^2 + \hat{q}^2) = \frac{\omega}{2}\frac{\hbar}{2}(\hat{a} + \hat{a}^\dagger)(\hat{a} + \hat{a}^\dagger) - \frac{\omega}{2}\frac{\hbar}{2}(a^\dagger - a)(a^\dagger - a)$$

$$= \frac{\hbar\omega}{4}\left(\cancel{\hat{a}\hat{a}} + \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger + \cancel{\hat{a}^\dagger\hat{a}^\dagger} - (\cancel{\hat{a}^\dagger\hat{a}^\dagger} - \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} + \cancel{\hat{a}\hat{a}})\right)$$

$$= \frac{\hbar\omega}{2}\left(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right)$$

normální uspořádání $\hat{a}^\dagger \; \hat{a}$ $\quad \hat{a}\hat{a}^\dagger = \hat{a}^\dagger\hat{a} + 1$

$$\boxed{\hat{H} = \hbar\omega\,\hat{a}^\dagger\hat{a} + \frac{\hbar\omega}{2}}$$

$$E = \langle\psi|\hat{H}|\psi\rangle = \hbar\omega\langle\psi|\hat{a}^\dagger\hat{a}|\psi\rangle + \frac{\hbar\omega}{2}$$

$E_0 \equiv 0$ stavový vektor základního stavu $|0\rangle$

$\langle 0|\hat{a}^\dagger\hat{a}|0\rangle = 0$ $\quad \boxed{\langle 0|0\rangle = 1}$

$|b\rangle = a|0\rangle$ $\quad \boxed{\hat{a}|0\rangle = 0}$

$\langle b|b\rangle = 0$

Báze vlastních stavů

Př: $|0\rangle$ ... vlastním stavem hamiltoniánu $\hat{H}$ s vlastní hodnotou 0 [až na $\frac{\hbar\omega}{2}$]

$$\hat{H}|0\rangle = 0|0\rangle = 0$$

$$\hbar\omega\,\hat{a}^\dagger\underbrace{\hat{a}|0\rangle}_{0} = 0 \qquad \text{Q.E.D.}$$

Př: $|1\rangle = \hat{a}^\dagger|0\rangle$ ... vlastním stavem $\hat{H}$ s energií $\hbar\omega$ [až na $\frac{\hbar\omega}{2}$]

$$\hat{H}|1\rangle = \hbar\omega\,\hat{a}^\dagger\hat{a}\hat{a}^\dagger|0\rangle = \hbar\omega\,\hat{a}^\dagger(\hat{a}^\dagger\hat{a}+1)|0\rangle = \hbar\omega\hat{a}^\dagger|0\rangle$$

$= \hbar\omega|1\rangle$

Q. E. D.

Př: Je $|1\rangle$ normovaný na jedničku?

$$\langle 1|1\rangle = \langle 0|a a^+|1\rangle = \langle 0|(a^+a+1)|0\rangle = \langle 0|0\rangle = 1$$

Q. E. D.

Co už známe?

$$\langle 0|\hat{H}|0\rangle = 0$$

$$\langle 1|\hat{H}|0\rangle = \hbar\omega\langle 1|a^+a|0\rangle = 0$$

$$\langle 0|\hat{H}|1\rangle = \hbar\omega\langle 0|a^+a|1\rangle = \hbar\omega(\overset{=0}{a|0\rangle})^+ a|1\rangle = 0$$

$$= \hbar\omega\langle 0|a^+\overset{a^+a+1}{aa^+}|0\rangle =$$

$$= \hbar\omega\langle 0|a^+a^+a + a^+|0\rangle$$

$$= \hbar\omega\langle 0|\underset{|1\rangle}{a^+|0\rangle} = 0$$

$$\langle 1|\underbrace{\hat{H}|1\rangle}_{\hbar\omega|1\rangle} = \hbar\omega\langle 1|a^+a|1\rangle = \hbar\omega = \hbar\omega\langle 0|aa^+aa^+|0\rangle = \dots$$

Další stavy harmonického oscilátoru

$$\hat{H} = \hbar\omega a^+a \quad \dots$$

$|0\rangle$; $a^+|0\rangle = |1\rangle$

Př: Ověříme normalizaci $|1\rangle$

$$\langle 1|1\rangle = \langle 0|aa^+|0\rangle = \langle 0|a^+a+1|0\rangle = 0 + \overset{=1}{\langle 0|0\rangle} = 1$$

Př: Je $a^+|1\rangle$ vlastním stavem $\hat{H}$?

$$|2'\rangle = a^+|1\rangle$$

$$\hat{H}|2'\rangle = \hbar\omega\, a^+ a a^+|1\rangle = \hbar\omega\,(\underbrace{a^+a^+a}_{a^+\hat{H}} + a^+)|1\rangle$$

$$= \hbar\omega|2'\rangle + \underline{\hbar\omega}\, a^+ \underline{a^+ a}|1\rangle$$

$$= \hbar\omega|2'\rangle + \hbar\omega|2'\rangle = 2\hbar\omega|2'\rangle$$

Q.E.D

Př: Norma stavu $|2'\rangle$

$$\langle 2'|2'\rangle = \langle 1|a a^+|1\rangle = \langle 1|a^+a + 1|1\rangle = \langle 1|1\rangle + \langle 1|1\rangle = 2$$

Def:

$$|2\rangle = \frac{1}{\sqrt{2}}\, a^+|1\rangle$$

Př: Dokažme, že $|n\rangle = \frac{1}{\sqrt{n!}}(a^+)^n|0\rangle$ je normovaným vlastním stavem $\hat{H}$ s vlastní energií $n\hbar\omega$.

Řešení:

$$|n\rangle \longrightarrow \hbar\omega\, a^+a|n\rangle = n\hbar\omega|n\rangle$$

$$\hbar\omega\, a^+a\, \underbrace{a^+|n\rangle}_{\sim|n+1\rangle} = \hbar\omega\, a^+(a^+a+1)|n\rangle =$$

$$= \hbar\omega\, a^+|n\rangle + \underbrace{\hbar\omega\, a^+(a^+a|n\rangle)}_{n\hbar\omega a^+|n\rangle}$$

$$= (n+1)\hbar\omega\, \underbrace{a^+|n\rangle}$$

Je-li $|n\rangle$ normovaný

$$\hat{n} = \hat{a}^\dagger \hat{a}$$

$$\langle n | a a^\dagger | n\rangle = ?$$

$$\hat{n}|n\rangle = n|n\rangle$$

$$\langle n | (a^\dagger a + 1) | n\rangle = n \underbrace{\langle n|n\rangle}_{=1} + 1 \underbrace{\langle n|n\rangle}_{=1} = n+1$$

$$\boxed{|n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle}$$

Maticová reprezentace operátorů harmonického oscilátoru

$$\hat{H} = \hbar\omega\, \hat{a}^\dagger \hat{a}$$

$$\{|n\rangle\} = \{|0\rangle, |1\rangle, |2\rangle, \dots\}$$

$$\hat{a}^\dagger, \hat{a} \,;\; \hat{\tilde{p}}, \hat{\tilde{q}}$$

$$|n+1\rangle = \frac{1}{\sqrt{n+1}} a^\dagger |n\rangle \Rightarrow a^\dagger |n\rangle = \sqrt{n+1}\, |n+1\rangle$$

$$\langle m | a^\dagger | n\rangle = \sqrt{n+1}\, \langle m | n+1\rangle$$

$$= \sqrt{n+1}\, \delta_{m,n+1}$$

$$\hat{a}^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 & & \\ 1 & 0 & 0 & 0 & & \\ 0 & \sqrt{2} & 0 & 0 & & 0 \\ 0 & 0 & \sqrt{3} & 0 & & \\ & & & \sqrt{4} & 0 & \\ & 0 & & & \ddots & 0 \\ & & & & & \ddots \end{pmatrix}$$

$$\hat{a} = (\hat{a}^\dagger)^\dagger = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & \sqrt{2} & & & 0 \\ & & 0 & \sqrt{3} & & \\ & & & 0 & \ddots & \\ & 0 & & & 0 & \ddots \\ & & & & & 0 \end{pmatrix}$$

$$\hat{H} = \hbar\omega\, \hat{a}^\dagger a \qquad \langle n|\hat{H}|m\rangle = \hbar\omega\, \langle n|a^\dagger a|m\rangle = m\hbar\omega\langle n|m\rangle = m\hbar\omega\,\delta_{nm}$$

$$\hat{H} = \begin{pmatrix} 0 & & & 0 \\ & \hbar\omega & & \\ & & 2\hbar\omega & \\ 0 & & & 3\hbar\omega \end{pmatrix}$$

$$\hat{q} = \sqrt{\frac{\hbar}{2}}\,(a + a^\dagger) =$$

$$\sqrt{\frac{\hbar}{2}} \begin{pmatrix} 0 & 1 & & & & & \\ 1 & 0 & \sqrt{2} & & & & \\ & \sqrt{2} & 0 & \sqrt{3} & & & \\ & & \sqrt{3} & 0 & \ddots & & \\ & & & \ddots & 0 & & \\ & & & & & 0 & \\ & & & & & & 0 \end{pmatrix}$$

$$\hat{p} = i\sqrt{\frac{\hbar}{2}}\,(a^\dagger - a) =$$

$$i\sqrt{\frac{\hbar}{2}} \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & -\sqrt{2} & & \\ & \sqrt{2} & 0 & -\sqrt{3} & \\ & & \sqrt{3} & 0 & \ddots \\ & & & \ddots & 0 \end{pmatrix}$$