Harmonický oscilátor v souřadnicové reprezentaci

$$H = \frac{\omega}{2}(p^2+q^2) \rightarrow \quad q = \sqrt{\frac{\hbar}{2}}\,(a^+ + a)$$

$$p = i\sqrt{\frac{\hbar}{2}}\,(a^+ - a)$$

$$\left.\begin{array}{l} p \rightarrow \frac{p}{\sqrt{\hbar}} = p \\ q \rightarrow \frac{q}{\sqrt{\hbar}} = q \end{array}\right\} \Rightarrow \quad H = \frac{\hbar\omega}{2}(p^2+q^2) \Rightarrow [\hat{p},\hat{q}] = -i$$

Souřadnicová reprezentace

$$\hat{q} = q \qquad |\psi\rangle \rightarrow \langle q|\psi\rangle \equiv \psi(q)$$

$$\hat{p} = -i\frac{\partial}{\partial q}$$

$$H = -\frac{\hbar\omega}{2}\frac{\partial^2}{\partial q^2} + \underbrace{\frac{\hbar\omega}{2}q^2}_{V(q)}$$

Základní stav (vlnová funkce základního stavu) $\psi_0(q)$

$$\tilde{H} = \hbar\omega a^+ a \quad \Rightarrow \quad \underline{\hbar\omega a^+ a|0\rangle = 0}$$

$$H = \hbar\omega a^+ a + \frac{\hbar\omega}{2}$$

$$\hbar\omega a^+ a = \frac{\hbar\omega}{2}\left(-\frac{\partial^2}{\partial q^2} + q^2 - 1\right) \Rightarrow \left(-\frac{\partial^2}{\partial q^2} + q^2 - 1\right)\psi_0(q) = 0$$

$$\psi_0(q) = A\, e^{-aq^2}$$

$$\frac{\partial}{\partial q}\psi_0(q) = -A\, 2aq\, e^{-aq^2} = -2aq\,\psi_0(q)$$

$$\frac{\partial^2}{\partial q^2}\psi_0(q) = -2a\,\psi_0(q) - 2aq\frac{\partial}{\partial q}\psi_0(q) = -2a\,\psi_0(q) + 4a^2q^2\psi_0(q)$$

$$-\left(-2a\,\psi_0(q) + 4a^2q^2\psi_0(q)\right) + q^2\psi_0(q) - \psi_0(q) = 0$$

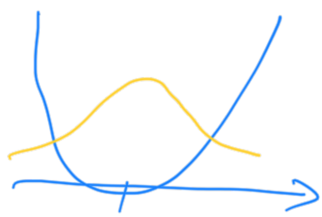
$$\left.\begin{aligned} 2a &= 1 \\ 4a^2 &= 1 \end{aligned}\right\} \quad a = \frac{1}{2}$$

$$\psi_0(q) = A\, e^{-\frac{q^2}{2}} \rightarrow \text{norma} \int_{-\infty}^{\infty} dq\, |\psi_0(q)|^2 = A^2\int_{-\infty}^{\infty} dq\, e^{-q^2}$$

$$= \sqrt{\pi}A^2 = 1$$

$$A = (\pi)^{-1/4}$$

$$\boxed{\psi_0(q) = \frac{1}{(\pi)^{1/4}}\, e^{-\frac{q^2}{2}}} \Rightarrow P_0(q) = \frac{1}{(\pi)^{1/2}}\, e^{-q^2}$$



$$\psi_n(q) = \langle q|\, \frac{a^\dagger}{\sqrt{n}}\, |n-1\rangle$$

$$a^\dagger = \frac{1}{\sqrt{2}}(\hat{q} - i\hat{p}) \Leftarrow \hat{p} = -i\frac{\partial}{\partial q}$$

$$= \frac{1}{\sqrt{2n}}\left(\langle q|\hat{q}|n-1\rangle - i\langle q|\hat{p}|n-1\rangle\right) =$$

$$= \frac{1}{\sqrt{2n}} \psi_{n-1}(q)\, q - \frac{i}{\sqrt{2n}} \langle q|\hat{p}|n-1\rangle$$

$$\hat{p} = \frac{i}{\sqrt{2}}(a^+ - a)$$

$$-i\langle q|\hat{p}|n-1\rangle = \frac{1}{\sqrt{2}}\langle q|(a^+ - a)|n-1\rangle = \frac{1}{\sqrt{2}}\Big(\langle q|\sqrt{n}\,\overbrace{\frac{a^+}{\sqrt{n}}|n-1\rangle}^{|n\rangle}$$

$$-\langle q|a|n-1\rangle\Big) = \sqrt{\frac{n}{2}}\psi_n(q) - \frac{1}{\sqrt{2}}\langle q|\underbrace{a\,a^+}_{a^+a+1}\frac{1}{\sqrt{n-1}}|n-2\rangle =$$

$$= \sqrt{\frac{n}{2}}\psi_n(q) - \frac{1}{\sqrt{2}}\frac{n-2}{\sqrt{n-1}}\psi_{n-2}(q) - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{n-1}}\psi_{n-2}(q)$$

$$= \sqrt{\frac{n}{2}}\psi_n(q) - \frac{1}{\sqrt{2}}\frac{n-1}{\sqrt{n-1}}\psi_{n-2}(q)$$

$$\psi_n(q) = \frac{q}{\sqrt{2n}}\psi_{n-1}(q) + \frac{1}{2}\psi_n(q) - \frac{1}{2}\sqrt{\frac{n-1}{n}}\psi_{n-2}(q)$$

$$\boxed{\psi_n(q) = \sqrt{\frac{2}{n}}\, q\, \psi_{n-1}(q) - \sqrt{\frac{n-1}{n}}\psi_{n-2}(q)}$$

$$\psi_{n+1}(q) = \sqrt{\frac{2}{n+1}}\, q\, \psi_n(q) - \sqrt{\frac{n}{n+1}}\psi_{n-1}(q)$$

<u>Ověření:</u>

$\psi_1(q)$ je vlastní funkce hamiltoniánu s energií

$3\frac{\hbar\omega}{2}$

$$\psi_1(q) = \sqrt{2}\, q\, \psi_0(q) \qquad ; \frac{\partial}{\partial q}\psi_1(q) = \sqrt{2}Ae^{-\frac{q^2}{2}} + \sqrt{2}Aq(-q)e^{-\frac{q^2}{2}}$$

$$= \sqrt{2}A(1-q^2)e^{-\frac{q^2}{2}}$$

$$\frac{\partial^2}{\partial q^2}\psi_1(q) = -\sqrt{2}A\,2q\,e^{-\frac{q^2}{2}} + \sqrt{2}(1-q^2)A(-q)e^{-\frac{q^2}{2}} =$$

$$= -\sqrt{2}A(3q-q^3)\,e^{-\frac{q^2}{2}}$$

$$\left(-\frac{\partial^2}{\partial q^2}+q^2\right)\psi_1(q) = \sqrt{2}A(3q-\cancel{q^3})\,e^{-\frac{q^2}{2}} + \cancel{q^2\sqrt{2}q}\,e^{-\frac{q^2}{2}} =$$

$$= 3\sqrt{2}Aq\,e^{-\frac{q^2}{2}} = 3\,\psi_1(q)$$

$$\hat{H}\psi_1(q) = \frac{3\hbar\omega}{2}\psi_1(q) \Rightarrow E_1 = \hbar\omega + \frac{1}{2}\hbar\omega$$